

Ginsparg-Wilson Games

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I implement a set of tricks for constructing lattice fermion actions which approximately realize the Ginsparg-Wilson relation, with very promising results from simulations.

It might be useful to have a simple lattice fermion action $S = \bar{\psi} D \psi$ which approximately obeys the Ginsparg-Wilson (GW) relation [1]

$$\{\gamma_5, D\} = D \gamma_5 R D. \quad (1)$$

Published algorithms[2] cost (apparently) hundreds of times as much as the usual clover action. I describe an approach which costs about a factor of $6.5 \times (N+1)$ as much as the clover action for an N th order approximation, and even $N = 1$ looks quite promising.

The ideas in this work are based on three remarkable formulas first published by Neuberger[3]: Introducing a zeroth-order Dirac operator D_0 and defining $z = 1 - D_0/r_0$, a GW action (with $R = r_0$) is

$$D_{GW} = r_0 \left(1 - \frac{z}{\sqrt{z^\dagger z}}\right). \quad (2)$$

The inverse square root is approximated by

$$\frac{1}{\sqrt{z^\dagger z}} \simeq \frac{1}{N} \sum_{j=1}^N \frac{1}{c_j z^\dagger z + s_j} \quad (3)$$

($c_j = \cos^2(\pi(j+1/2)/(2N))$, $s_j = 1 - c_j$) and

$$1 - \frac{A}{B} = \frac{B - A}{B}. \quad (4)$$

Here $B - A = W^{(N)}$ is the polynomial

$$\prod_{j=1}^N (c_j z^\dagger z + s_j) - (z/N) \sum_j \prod_{i \neq j} (c_i z^\dagger z + s_i). \quad (5)$$

To use this for propagators, note $D_{GW}^{(N)} \psi = \phi$ is

$$\psi = (D_{GW}^{(N)})^{-1} \phi = B(W^{(N)})^{-1} \phi. \quad (6)$$

(i.e. ψ is found by inverting the simple differential operator W , and then multiplying by the local operator B .) Of course, one needs a D_0 for which Eqn. 3 works well for small N .

A good D_0 should already be very chiral. This immediately suggests that we begin with a fat link action—these actions are already quite chiral as shown by their small mass renormalization and $Z_A \simeq 1$) [4].

The eigenvalues of a GW action lie on a circle. I determine the best D_0 by taking a free field test action and varying its parameterization to optimize its eigenvalue spectrum (in the least-squares sense) for circularity, for some r_0 (which can also be varied; the optimal value is about 1.6). The action of choice is "planar:" it has scalar and vector couplings $S = \sum_{x,r} \bar{\psi}(x)(\lambda(r) + i\gamma_\mu \rho_\mu(r))\psi(x+r)$ for r connecting nearest neighbors ($\vec{r} = \pm\hat{\mu}$; $\lambda = \lambda_1 = -0.170$, $\rho_\mu = -0.177$) and diagonal neighbors ($\vec{r} = \pm\hat{\mu} \pm \hat{\nu}$, $\nu \neq \mu$; $\lambda = \lambda_2 = -0.061$, $\rho_\mu = -0.053$; $\lambda(r=0) = -8\lambda_1 - 24\lambda_2$). The approach of the eigenvalues to a circle is shown in Fig. 1. The massive action for bare mass m is obtained from the $m = 0$ one by[5] $D(m) = (1 + am/2)D_0 + am$.

One might think that the iteration could be done starting with the Wilson or clover action. The trick of Eqn. 3 does rapidly pull the eigenmodes onto a circle, but the problem is the decomposition into the W/B form. Unlike for the planar action, the eigenvalues of W are thrown far out into the complex plane. This is shown in Fig. 2. Unfortunately, W is the matrix which is to be inverted for propagators. Since the high momentum modes of a fat link action don't see the gauge fields very well, they behave like free field modes.

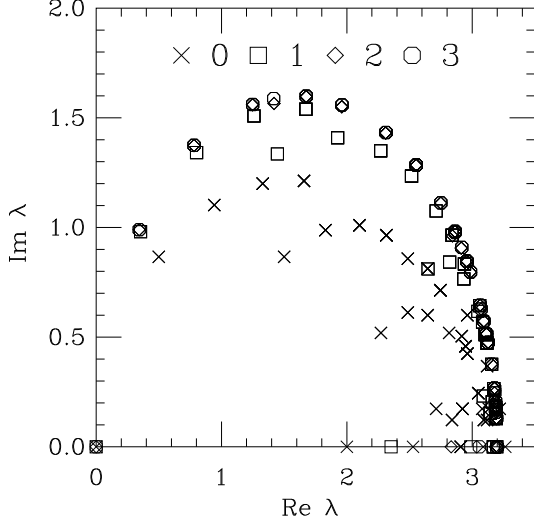


Figure 1. Free field spectrum of $D_{GW}^{(N)}$, with the planar action as the kernel, for $N = 0, 1, 2, 3$. Only the $\text{Im } \lambda > 0$ eigenvalues are shown.

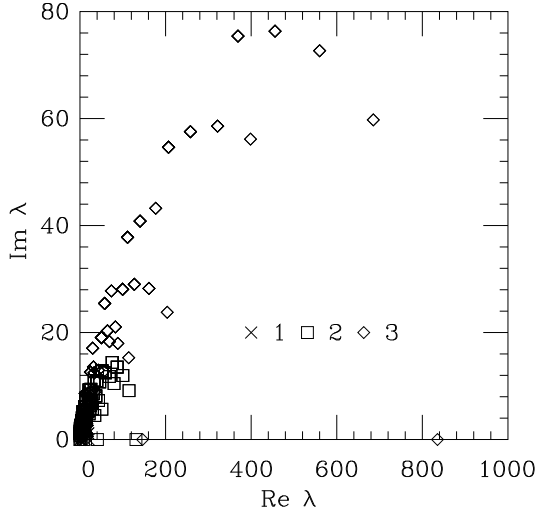


Figure 2. Free field spectrum of the operator $W^{(N)}$, with the Wilson action as the kernel, for $N = 1, 2, 3$.

The wide spread of eigenvalues means that in real simulations, $W^{(N)}$ becomes ill-conditioned even for small N .

Chiral properties of the action, in four dimensions, are tested first by computing the value of its smallest real eigenvalue λ on a set of isolated instanton configurations (the instanton radius is ρ) (Fig. 3). In an exact GW action the real eigenvalue would be zero until the instanton fell through the lattice, when it would disappear. In an ordinary action, λ is a smooth function of ρ , close to zero for big ρ and moving away from zero, generally to a positive value, as ρ decreases, until the eigenvalue collides with a doubler and goes imaginary. For a better action, λ keeps closer to zero and breaks away more steeply, with a step function for λ as the desired limiting result.

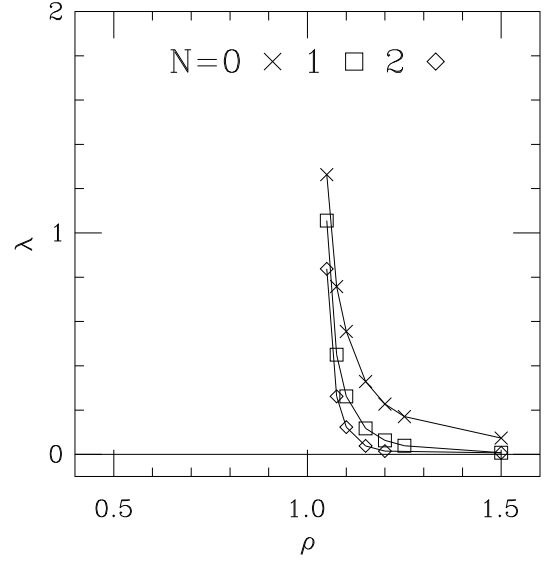


Figure 3. Real eigenvalue spectrum of $D_{GW}^{(N)}$ on background instanton configurations, for $N = 0, 1, 2$.

As shown by the pion mass in Fig. 4 (quenched, for $SU(3)$, $a = 0.2$ fm, $8^3 \times 24$ lattice), the zeroth order action is already very chiral and $N = 1$ iteration is even more so.

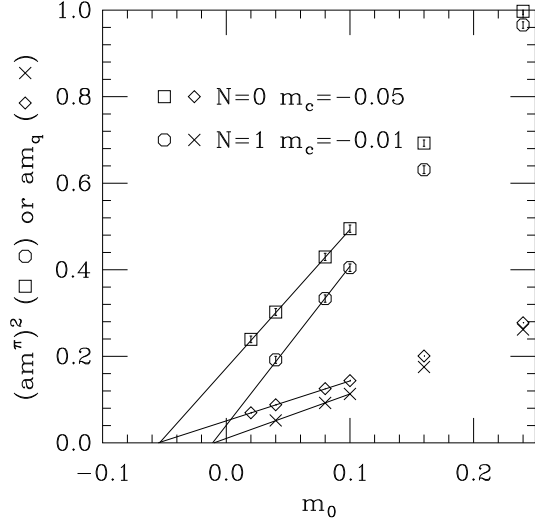


Figure 4. Pion mass and quark mass (from the PCAC relation) for $N = 0, 1$.

In Fig. 5 I show the N/ρ mass ratio at $\pi/\rho = 0.7$ for the $N = 0$ and 1 versions of this action, along with other actions. Both of the new actions (on improved background gauge configurations) have small scaling violations for this observable.

As an added feature, the hadron dispersion relation for these actions is better than for the clover action (the extra terms in the planar action can be tuned to optimize this[6]).

Finally, a rough calculation of Z_A from Ward identities produces a value quite close to unity—as the fat link clover action[4] gave.

To conclude: this is an approach towards the construction of a GW action in which all the cost is “up-front” in the evaluation of D_0 , but the gain is that probably only a few terms (maybe just $N = 0$ or 1) in the expansion of $D_{GW}^{(N)}$ are needed. One also only needs to invert the simple (but messy) differential operator $W^{(N)}$ (no inverse inside an inverse is needed). It would be very interesting to tackle the hard lattice problem of $\langle \bar{\psi}\psi \rangle$ along the lines of Ref. [7] with this approach.

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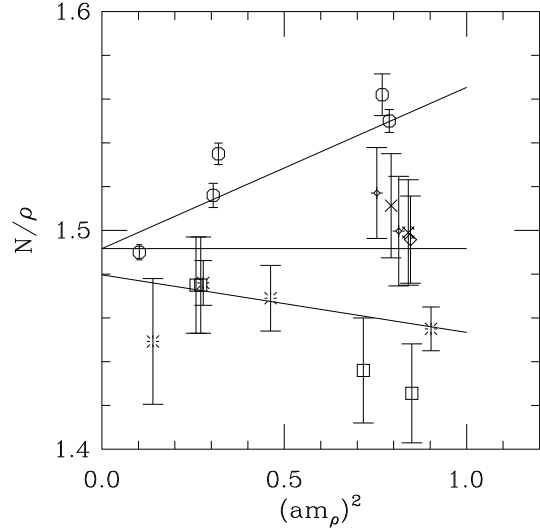


Figure 5. N/ρ mass ratio at $\pi/\rho = 0.7$. $N = 0$ and $N = 1$ actions at $a = 0.2$ fm are fancy crosses, octagons—KS fermions, bursts are NP C_{SW} thin link actions, squares fat link clover on Wilson gauge configurations, and the rest on improved gauge configurations.

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